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EXPEDIENT TRANSFORMATIONS IN STRUCTURAL MECHANICS

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ABSTRACT

In this paper some topological transformations are designed for simplifying certain problems involved in mechanics of structures. For each case, the main problem is stated and the proposed topological transformation is established. Once the required topological analysis is completed, a back transformation results the solution for the main problem. Expedient transformations studied here employ (1) models drawn on a plane, (2) models embedded into higher dimensional spaces, (3) interchange models defined which have more simple connectivity properties than the corresponding original structural model.

Keywords: transformation, graph theory, topology, rigidity, analysis, force method, ordering, decomposition, configuration processing

1. INTRODUCTION

Analysis of systems and in particular structures can be decomposed into three phases:

1. Approximation, followed by choosing an appropriate model.
2. Specifying topological properties followed by a topological analysis.
3. Assigning algebraic variables, followed by an algebraic analysis. Such a decomposition results in a considerable simplification in the analysis and leads to a clear understanding of the structural behaviour.

For an optimal analysis of a structure, three conditions should be fulfilled. The structural (stiffness or flexibility) matrices should be sparse, properly structured (e.g. banded) and well-conditioned. The latter property is not purely topological and is treated elsewhere, Kaveh [1]. Only problems relevant to sparsity and proper structuring are studied in this paper.

Pattern equivalence of structural matrices and those of graph theory simplifies structural problems and allows advances made in this field to be transferred to structural mechanics. As an example, for rigid-jointed frames the sparsity of flexibility matrices can be provided by construction of sparse cycle adjacency matrices. Similarly using sparse cut set bases, the formation of sparse stiffness matrices become feasible. Proper structuring of the flexibility and

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stiffness matrices of a structure can also be provided by structuring the pattern of cycle and cut set adjacency matrices of its model, respectively.

This paper is devoted to the study of some structural problems in which topological graph theory plays an important role. Topological graph theory is primarily concerned with representing graphs on surfaces. An embedding or a drawing of a graph can be considered as identification of the graph with a subset of a surface in an appropriate fashion. For some problems it is beneficial to define a new graph with more simple connectivity properties than the original model.

Some of the mathematical definitions used in this paper are presented in the Appendix. For further concepts and definitions the reader may refer to the author's recent book, Kaveh [2].

2. DEGREE OF STATIC INDETERMINACY OF SPACE STRUCTURES

The first step in the analysis of a structure by means of the force method consists of determining its degree of indeterminacy (DSI). For space structures, an efficient approach can be developed by drawing the structural model on the plane, using the two simple theorems presented in this section.

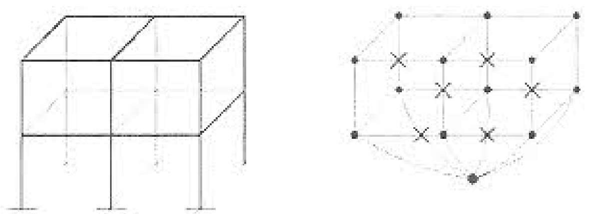
Definitions: A *drawing* S^P of S is a mapping of S into a surface. The nodes of S go into distinct nodes of S^P. A member and incidence nodes map into a homeomorphic image of the closed interval [0,1] with the relevant nodes. A *good drawing* is one in which no two members are incident with a common point, and no two members have more than one point in common. A common point of two members is a *crossing*. An *optimal drawing* in a given surface is one which exhibits the least possible crossing. The number of crossing points of S after drawing on a plane or a sphere, S^P, is denoted by $v(S^P)$ For cases when the drawing is optimal, $v(S^P)$ becomes the *crossing number* of the graph S.

Theorem A: For a space frame S the degree of static indeterminacy is given by

$$\gamma(S) = 6b_1(S^P) = 6[R_i(S^P) - v(S^P)], \tag{1}$$

where $R_i(S^P)$ is the number of internal regions of S^P: i.e.

$$R_i(S^P) = R(S^P) - 1. \tag{2}$$



(a) A space frame S (b) A drawing S^P of S.

Figure 1. A space frame with an arbitrary drawing

Example - For a space frame depicted in Figure 1(a), a drawing may be considered as shown in Figure 1(b). Using Eq. (1) results in

$$\gamma(S) = 6[20 - 6] = 84.$$

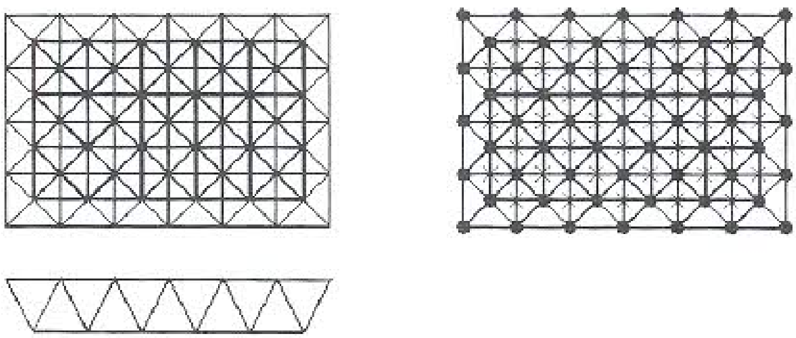
Theorem B: For a space truss the degree of static indeterminacy is given by

$$\gamma(S) = \nu(S^P) - M_c(S^P), \tag{3}$$

where $M_c(S^P)$ is the number of members required for full triangulation of S^P.

Example - A space truss S supported in a statically determinate fashion together with a drawing S^P of S are shown in Figure 2. Employing Eq.(3) leads to

$$\gamma(S) = 38 - 17 = 21.$$



(a) A double layer grid S (b) A arbitrary drawing of S

Figure 2. A space truss S and an arbitrary drawing of S

Simple proofs of the above two theorems may be found in Kaveh [3].

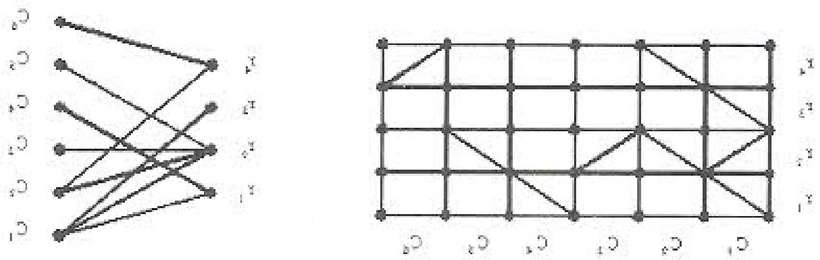
Naturally it is advantageous to use optimal drawings in order to reduce the number of countings for calculating $\gamma(S)$ of structures. An optimal drawing of a structure with zero crossing number has an attractive property, since the cycles bounding the finite regions of the drawing form a suitable basis, known as a mesh basis.

3. RIGIDITY OF GRID-SHAPED PLANAR TRUSSES; A BIPARTITE GRAPH

The study of the rigidity of planar trusses is due to Laman [4], who found the necessary and sufficient conditions for the rigidity of this type of structures. Lovasz and Yemini [5] and Sugihara [6] developed algorithms for controlling the rigidity. There is a special type of planar trusses for which the rigidity can be checked more efficiently, Bolker and Crapo [7]. In the latter approach, a bipartite graph is defined for the truss and the connectedness of this graph results in the rigidity of the main truss.

Consider a grid-shaped planar truss with m rows and n columns with some diagonal bracings. Associate one vertex with each row and one vertex with each column. Connect a row vertex to a column vertex if the corresponding panel has a diagonal member, resulting in a bipartite graph B(S). It can easily be proved that S is rigid if B(S) is connected. Furthermore at least m + n - 1 diagonal members are needed for minimal rigidity of S.

Example - A grid-shaped planar truss is shown in Figure 3a. The bipartite graph B(S) of S is illustrated in Figure 3b. Since B(S) is connected, therefore S is rigid. It can be seen that the removal of any diagonal bracing member of S will result in a disconnected B(S), destroying the rigidity of S.



(a) A planar truss S (b) The bipartite graph B(S) of S

Figure 3. A grid-shaped planar truss and its bipartite graph

4. CYCLE BASES SELECTION; MANIFOLD EMBEDDING

The force method of frame analysis requires the formation of suitable statical bases corresponding to sparse flexibility matrices. Due to the pattern equivalence of a flexibility matrix of a frame and the cycle adjacency matrix of its graph model, the problem can be transformed to the formation of a maximal set of independent cycles, known as a *cycle basis*, Kaveh [8-12]. In order to have a sparse flexibility matrix for a frame structure whose elements have the least overlaps should be selected (*optimal cycle basis*). The formation of an optimal cycle basis is not simple, however, such a basis is quite often in-between cycle bases of the least length (*minimal cycle basis*). There are various methods for the selection of subminimal cycle bases, some of which will be described in the next three sections.

The process of embedding a graph S on a union of disks can be summarized as follows:

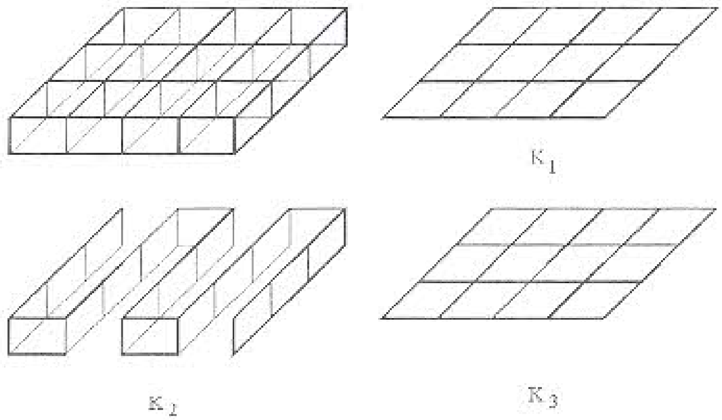
Step 1. Identify a planar subgraph and embed it on the first disk whose dissection is isomorphic to the selected subgraph.

Step 2. Select the second subgraph such that the corresponding dissection has a 2-cell with a free 1-face, and its intersection with the previous dissection is a connected subspace of the frontier of the first disk.

Step k. Repeat the process of the second step, identifying the *ith* planar subgraph, whose dissection has a 2-cell with a free 1-face and the intersection of its dissection with the previously selected dissections is a connected subspace of the frontier of the *ith* disk.

Repeat step k until all members (1-cells) of S are embedded, and the regional cycles forming a cycle basis is obtained.

Example: S is a space graph as shown in Figure 4a, which is embedded on the union of three disks K_1, K_2 and K_3 as depicted in Figure 4b.



(a) a space graph S (b) the selected disks

Figure 4. A space graph and the identified disks

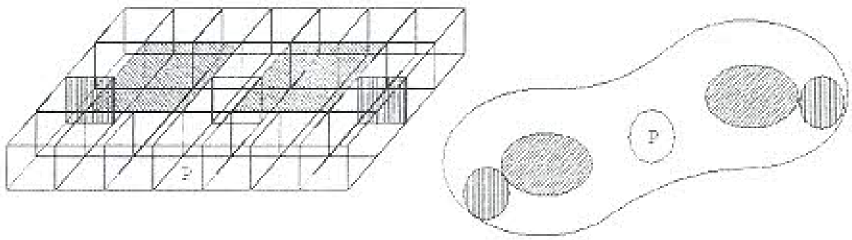
In order to reduce the overlaps of the selected cycles, it is ideal to embed S on a minimum number of disks. This number is known as the *thickness* of the graph. However, the restrictions imposed, and the lack of efficiency of the available methods for embedding, reduces the chance of the minimality of such an embedding, Kaveh [2].

5. CYCLE BASES SELECTION; A MANIFOLD EMBEDDING

The planar embedding of a graph, results in a set of independent regional cycles forming a *mesh basis* . However, for non-planar graphs other embeddings should be employed. A manifold embedding is an example of this kind. A cycle basis can be obtained by embedding S on an admissible manifold, Henderson and Maunder [13]. An orientable manifold may be viewed as an sphere with "h" handles. In order to guarantee the independence of the corresponding regional cycles, 2h fillings and one perforation of order 2 should be made, resulting in an admissible manifold embedding.

Example - A hollow box S is considered as shown in Figure 5a, which is embedded on a sphere with one handle. Therefore two fillings (shaded) and one perforation have been considered, Figure 5b. The selected cycle basis consists of 79 four-sided cycles and two eight-sided cycles.

In manifold embedding the quality of the selected cycle basis depends on the number of handles being used. It is ideal to embed S on a sphere with minimum number of handles. This number is known as the *genus* of the graph. Again there is no efficient method for such an embedding, Kaveh [14].

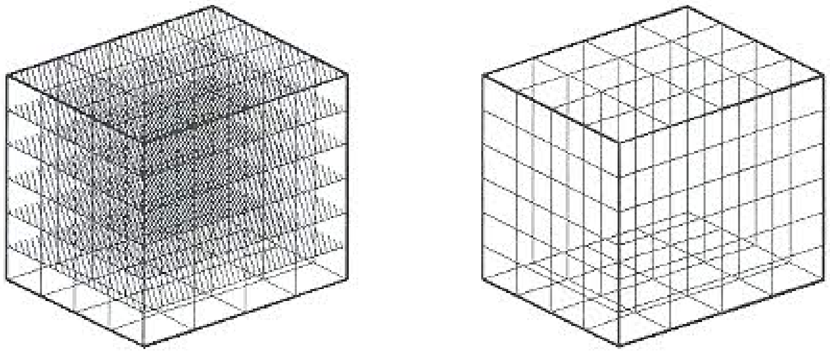


(a) A space structure S (b) A manifold embedding
Figure 5. An admissible manifold embedding of S

6. CYCLE BASES SELECTION; COLLAPSIBLE EMBEDDING

A graph can be viewed as the 1-skeleton of a 3-complex. An n-cell is called *collapsible* if it can be shrunk into the reminder of its n-1 cells through a free n-1 cell. If a 3-complex can be collapsed into a point, then it is called *collapsible*. It can be proved that a collapsible 3-complex can be used for the formation of a cycle basis of its 1-skeleton. This can be achieved by collapsing all the 3-cells through free 2-cells or 2-cells being freed in subsequent steps, Maunder [15].

Example - Consider a space structure as shown in Figure 6(a). This graph can be viewed as the 1-skeleton of a 3-complex as depicted in Figure 6(b). After collapsing all the 3-cells through the shaded 2-cells, the bounding cycles of the remaining 2-complex ($16+10\times24=256$ cycles) form a cycle basis of S.



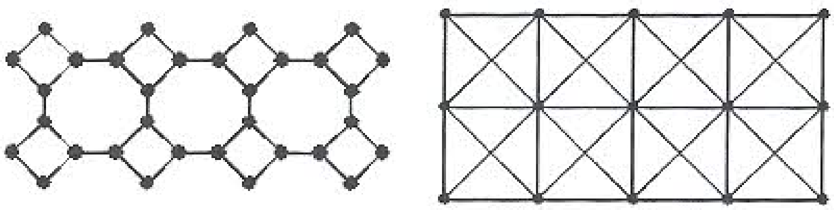
(a) A space structure S (b) S embedded on a 3-complex
Figure 6. A space structure and its collapsible embedding

7. GENERALIZED CYCLE BASES; INTERCHANGE GRAPH

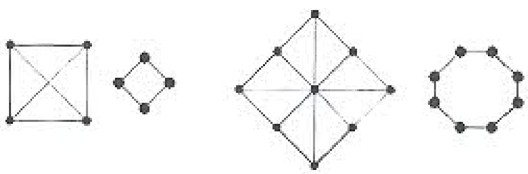
For a general skeletal structure, a statical basis can be formed on a maximal set of subgraphs defined as a *generalized cycle basis* (GCB) of S, Kaveh [16]. Such a basis has been defined as the consequence of generalizing the first Betti number $b_1(S) = M(S) - N(S) + b_0(S)$ to $\gamma_1(S) = aM(S) + bN(S) + c\gamma_0(S)$. The formation of a generalized cycle basis, in general, can be time consuming. However, for planar trusses the problem can be simplified by using a special graph, known as the *interchange graph*. An interchange graph I(S) of S is a graph whose vertices are in a 1-to-1 correspondence with the triangular regions of S (when S is embedded in the plane) and two nodes are connected by an edge if the corresponding triangles have a common member.

In order to form a generalized cycle basis of S, one can generate a cycle basis of I(S), and with a back transformation the elements of the generalized cycle basis can then be obtained, Kaveh [13].

Example - A planar truss as shown in Figure 7(a) is considered. The interchange graph of S is formed as depicted in bold lines in the same figure. A cycle basis of I(S) consists of 11 regional cycles leading to 18 subgraphs forming a GCB of S. On each subgraph one self-equilibrating stress system (S.E.Ss) can be constructed, corresponding to a suitable statical basis.



(a) A planar truss (b) The interchange graph



(c) Typical elements of the GCB

Figure 7. A planar truss and typical elements of the selected GCB

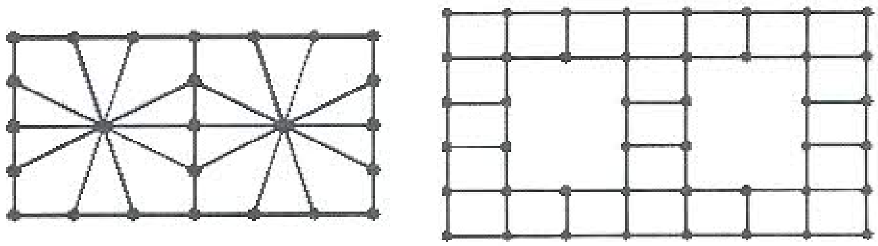
The regions of S after embedding in the plane does not need to be all triangulated. For such models, however, different types of cycles for I(S) should be employed, Kaveh [2].

8. CYCLE AND GENERALIZED CYCLE BASIS ORDERING; AN ASSOCIATE GRAPH

In order to reduce the bandwidth of the flexibility matrix of a structure, the bandwidth of its generalized cycle basis adjacency matrix can be reduced. For this purpose the *associate graph* of the selected basis should be constructed. Such a graph has its vertices in a 1-to-1 correspondence with the elements of the selected basis, and two vertices are connected by an edge if they have a member in common. This graph can also be used for ordering the elements of a null basis employed in algebraic force method, Kaneko et al [17].

Example - Let S be a planar graph as shown in Figure 8(a). Using the author's cycle selection algorithm [18], the following cycles are selected as a basis:

The associate graph of this basis is depicted in Figure 8(b). Using a nodal ordering algorithm (see for example Kaveh [19]) the node of A(C), hence the order of the cycles is obtained. Forming three S.E.Ss on each cycle yields a statical basis corresponding to a banded flexibility matrix. It should be noted that the selected cycles of a graph need not be regional cycles mesh basis), and the associate graph can easily be considered for any other type of cycle basis.



(a) A simple graph S (b) The associate graph of the cycle basis

Figure 8. Graph S and the associate graph of the selected cycle basis

9. GRAPH MODELS OF FINITE ELEMENT MESHES

In order to transform the nodal numbering of a finite element mesh into the graph nodal ordering, ten algorithms are presented in this section, Refs [3-4].

9.1. Element clique graph method (ECGM)

Definition: The *element clique graph* S of a FEM, is a graph whose nodes are the same as those of the FEM and two nodes n_i and n_j of S are connected with a member if n_i and n_j belong to the same element in the FEM.

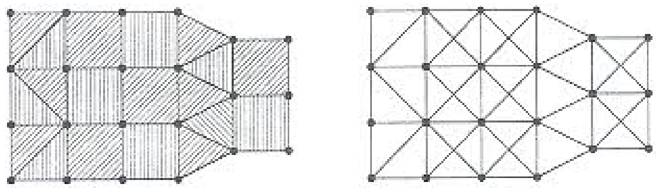


Figure 9. A FEM and its element clique graph.

9.2. Skeleton graph method (SKGM)

Definition: The 1-skeleton graph (*skeleton graph*) S of a FEM, is a graph whose nodes are the same as those of the FEM, and its members are the members of the FEM.

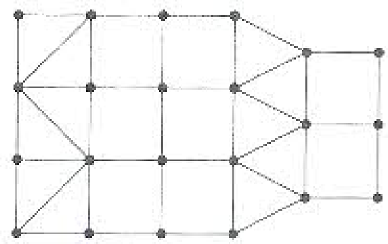


Figure 10. The skeleton graph of the FEM

9.3. Element star graph method (ESGM)

Definition: The *element star graph* S of a FEM has two set of nodes; namely the main set containing the same nodes as those of the FEM and the virtual set consisting of the virtual nodes in a one-to-one correspondence with the elements of the FEM. The member set of S is constructed by connecting the virtual node of each element i to all nodes of the element i.

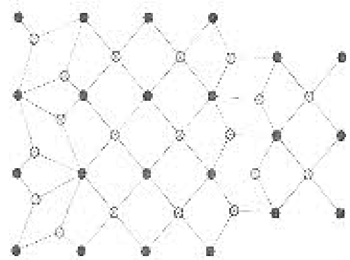


Figure 11. The element star graph of the FEM

9.4. Element wheel graph method (EWGM)

Definition: The *element wheel graph* S of a FEM is the union of the element star graph and the skeleton graph of the FEM. The element wheel graph of the FEM shown in Figure 9(a) is illustrated in Figure 12. The virtual nodes are shown by bigger dots.

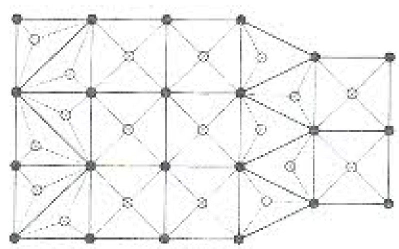
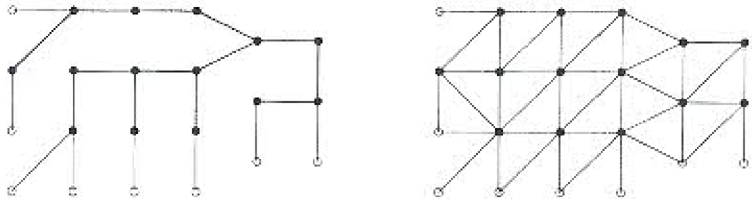


Figure 12. The element wheel graph of the FEM

9.5. Partially triangulated graph method (PTGM)

Definition: The *partially triangulated graph* S of a FEM is a graph whose nodes are the same as those of the FEM and a selected node of each element i is connected to all the nodes of i.



(a) The skeleton graph and an SRsubtree of the FEM (b) The partially triangulated graph

Figure 13. The partially triangulated graph of the FEM

9.6. Triangulated graph method (TRGM)

Definition: The *triangulated graph* S of a FEM is the union of the partially triangulated graph and the skeleton graph of the FEM.

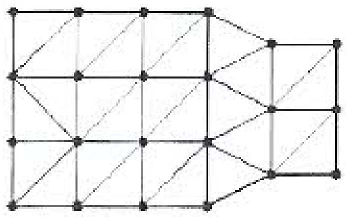


Figure 14. The triangulated graph of the FEM

9.7. Natural associate graph method (NAGM)

Definition: The *natural associate graph* S of a FEM has its nodes in a one-to-one correspondence with elements of the FEM, and two nodes of S are connected by a member if the corresponding elements have a common boundary. The natural associate graph of the FEM shown in Figure 9(a) is illustrated in Figure 15.

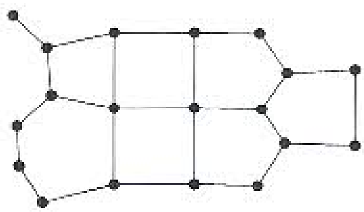


Figure 15. The natural associate graph of the FEM

9.8. Incidence graph method (INGM)

Definition: The *incidence graph* S of a FEM has its nodes in a one-to-one correspondence with the elements of the FEM, and two nodes are connected with a member if the corresponding elements have a common node.

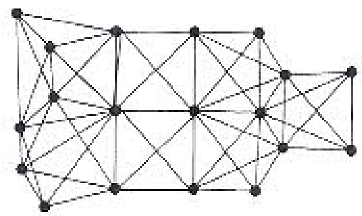


Figure 16. The incidence graph of the FEM

9.9. Representative graph method (REGM)

Definition: Consider the skeleton graph and select an appropriate starting node, using any algorithm available (e.g. an algorithm of Refs [3-4]). The nearest corner node of each element of the FEM is taken as the representative node of that element. The SRsubtree of the skeleton graph of the FEM containing all representative nodes of the elements is called a *representative graph* S of the FEM.

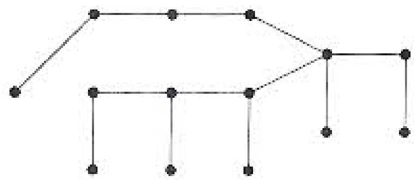


Figure 17. The representative graph of the FEM

9.10. Complete representative graph method (REGM)

Definition: This graph is the same as REG with additional members connecting each pair of nodes in CREG if their corresponding nodes in the FEM are contained in the same element.

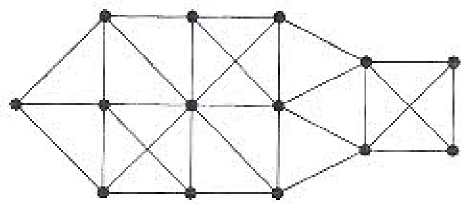
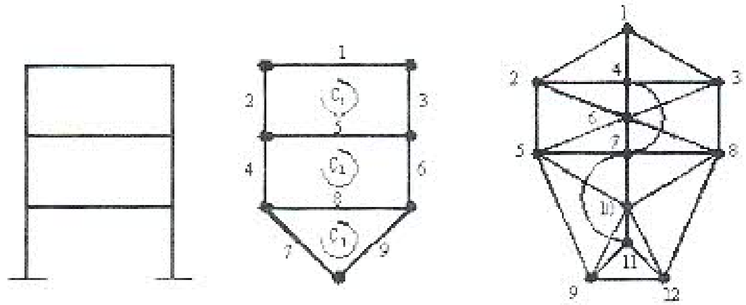


Figure 18. The complete representative graph of the FEM



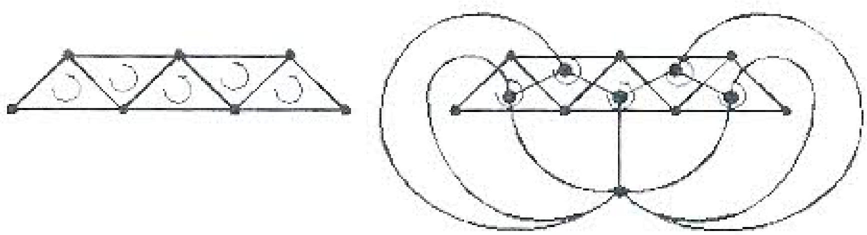
(a) S and the selected cycles (b) K-total graph of S
Figure 20. Member and cycle ordering of a graph

12. DUALITY OF CYCLE BASES AND CUT SET BASES; A DUAL GRAPH

The dual graph of a graph has many applications in mathematics and engineering. The *dual graph* D(S) of a planar graph has its vertices in a one-to-one correspondence with the regions of S (when embedded in the plane) and two vertices are connected with an edge of D(S) if there is a common member in their boundary. Naturally the number of edges and vertices of D(S) are the same as the numbers of members and regions of S, respectively. D(S)

A cycle basis of S corresponds to a cut set basis of D(S) and vice versa. This property makes the efficient generation of one basis from the other one feasible by a simple transformation.

Example- A planar graph S is shown in Figure 21(a) with its dual graph given in bold lines. A typical cycle of S and the corresponding cut set are depicted in Figure 21(b). The duality for the graph model of a planar truss and its Maxwell diagram is an interesting problem in structural mechanics.



(a) A planar graph S and its dual (b) Typical cycle and cut set
Figure 21. A planar graph and its dual graph

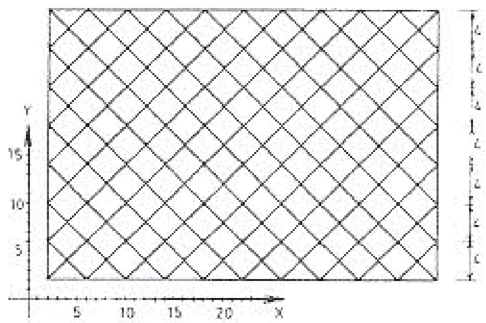
13. CONFIGURATION PROCESSING

The mathematical models of a practical structure can often be generated by using its repeated units employing translation, rotation, deflection or combinations of these functions. Such a unit can easily be represented in an integer coordinate system. Once the whole model is formed, a simple transformation can map the generated model into the structural model containing its geometrical information (Kaveh [2] and Nooshin [23]).

Example - Consider a grid S as shown in Figure 22(a), generated in an integer coordinate system and transformed to real coordinate system b the following transformation:

$$x = 2I_1$$

and $$y = 2I_2$$



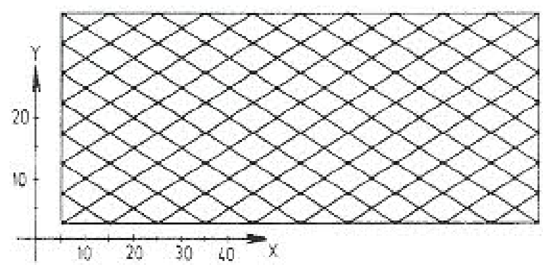
(a) A planar grid S (Ref. [23])

Using the transformation as

$$x = 5I_1$$

and $$y = 2.5I_2$$

leads to the following configuration, Figure 22(b).

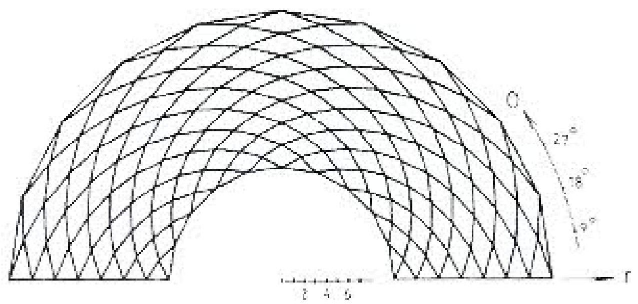


(b) A scaling transformation

Now apply the following functions using a polar coordinate system:

$$\theta = (I_1 - 1)\frac{\pi}{20} \quad \text{and} \quad r = I_2 + 9.$$

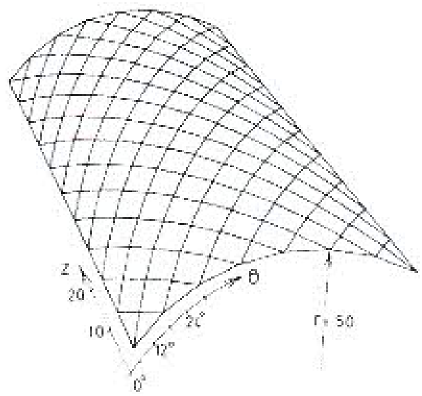
This leads to a grid as shown in Figure 22(c).



(c) A polar coordinate transformation of S

Similarly applying the following transformation leads to a barrel type of space structure as shown in Figure 22(d). Ref. [23].

$$r = 50, \ \theta = I_2 \frac{\pi}{30} \text{ and } z = 4I_1$$



(d) A cylindrical transformation of S
Figure 22. A grid and its transformations [23]

14. CONCLUDING REMARKS

A collection of topological transformations is presented for the study of topological properties of structures. These transformations provide useful tools for optimal analysis of structures, however, not all the transformations necessarily provide the best possible solution for the corresponding problem. As an example, a suitable cycle basis of a graph for the flexibility analysis can more efficiently be generated using the author's expansion process, [2,8]. It is hoped that other transformations can be found and better classifications can be made.

REFERENCES

1. Kaveh, A., "Optimizing the conditioning of structural flexibility matrices", *Computers and Structures*, **41**(1991) 489-494.
2. Kaveh, A., Structural Mechanics: Graph and Matrix Methods, *Research Studies Press (John Wiley)*, 2nd Edition, UK, 1995.
3. Kaveh, A., Optimal Structural Analysis, *Research Studies Press (John Wiley)*, UK, 1997.
4. Laman, G., "On graphs and rigidity of plane skeletal structures", *Journal of Engineering Mathematics*, 4(1970)331-340,.
5. Lovasz, L. and Yemini, Y., "On generic rigidity in the plane", *SIAM Journal of Discrete Methods*, **3**(1982)91-98.
6. Sugihara, K., "On some problems in the design of skeletal structures", *SIAM Journal of Discrete Methods*, **4**(1983)355-362.
7. Bolker, E.D. and Crapo, H., "How to brace a one storey building", *Environmental Planning B*, **4**(1977)125-152.
8. Kaveh, A., Application of Topology and Matroid Theory to the Flexibility Analysis of Structures, Ph.D. thesis, *London University, Imperial College of Science and Technology*, 1974.
9. Kaveh, A., "Improved cycle bases for the flexibility analysis of structures", *Computer Methods Applied Mechanical Engineering*, **9**(1976)267-272.
10. Kaveh, A., "Recent development in the force method of structural analysis", *Applied Mechanics Review*, **45**(1992)401-418.
11. Kaveh A., "Suboptimal cycle bases of graphs for the flexibility analysis of skeletal structures", *Computer Methods Applied Mechanical Engineering*, **71**(1988)259-271.
12. Henderson, J.C. de C. and Maunder, E.A.W., "A problem in applied topology", *Journal of the Institution of Mathematics and its Applications*, **5**(1969)245-269.
13. Kaveh, A., "Crossing number, genus and thickness of a graph for the flexibility analysis of structures", *Trends in Applications of Mathematics to Mechanics*, Edited by Troger et al, *Longman*, London, 1991.
14. Maunder, E.A.W., "Topological and linear analysis of skeletal structures" Ph.D. thesis, *London University*, IC, 1971.
15. Kaveh, A., "Statical bases for efficient flexibility analysis of planar trusses", *Journal of Structural Mechanics*, **14**(1986)475-488.
16. Kaneko, I. Lawo, M. and Thierauf, O., "On computational procedure for the force method", *International Journal of Numerical Methods in Engineering*, **18**(1982)1460-1495.
17. Kaveh, A., "An efficient program for generating cycle bases for the flexibility analysis of structures", *Communications in Applied Numerical Methods in Engineering*, **2**(1986)339-344.
18. Kaveh, A., Ordering for bandwidth reduction", *Computers and Structures*, **24**(1986)413-420.
19. Fenves, S.J. and Law, K.H., "A two - step approach for finite element ordering", *International Journal of Numerical Methods in Engineering*, **19**(1983)891-911.
20. Kaveh, A., "A note on a two-step approach for finite element ordering", *International Journal of Numerical Methods in Engineering*, **20**(1984)1553-1554.
21. Kaveh, A., "Bandwidth reduction of rectangular matrices", *Communications in Applied Numerical Methods in Engineering*, **9**(1993)259-267.
22. Nooshin, H., Formex Configuration Processing in Structural Engineering, *Elsevier Applied Science Publishers*, London, 1984.

APPENDIX; DEFINITIONS FROM GRAPH THEORY

In order to describe the concepts and methods of this paper in a self-contained manner, a number of definitions are presented in the following:

A *graph* S consists of a set of elements called *nodes* (vertices) and a set of elements called *members* (edges), together with a relation of incidence, associating two distinct nodes with each member. A graph is called *connected* if every pair of its nodes is joined together by a path. A *subgraph* of S is a graph having all its nodes and members in S. Two nodes of S are called *adjacent* if these nodes are the end nodes of a member. A member is *incident* to a node if the node is an end node of the member.

A *path* is a finite sequence of alternately distinct nodes and members of the graph. A path becomes a *cycle* if the first node and the last node of the path coincide. A *cut set* is a set of members of S such that the removal of these members from S results in a disconnected graph.

A maximal set of independent cycles (cut sets) is known as a *cycle (cut set) basis* of S. The cardinality of a cycle basis is the same as the first Betti number $b_1(S) = M(S) - N(S) + b_0(S)$ of S, where M(S), N(S) and $b_0(S)$ are the number of members, nodes and components of S, respectively. A *cycle adjacency matrix* is a $b_1(S) \times b_1(S)$ matrix consisting of 0 and 1 entries. An entry is 1 if the corresponding cycles have at least a member in common and 0 otherwise. A *cut set adjacency matrix* has $N(S) - b_0(S)$ columns and rows, and is defined analogously.

A graph is called *planar* if it can be embedded in the plane with no members crossing each other. A bipartite graph consists of two sets of nodes A and B such that only the nodes of A are joined to the nodes of B by members of the graph. A graph is called *clique* if all of its nodes are connected to each other.